4. Theory of Fraunhofer Diffraction

4.1 Fraunhofer Diffraction of Single Slit

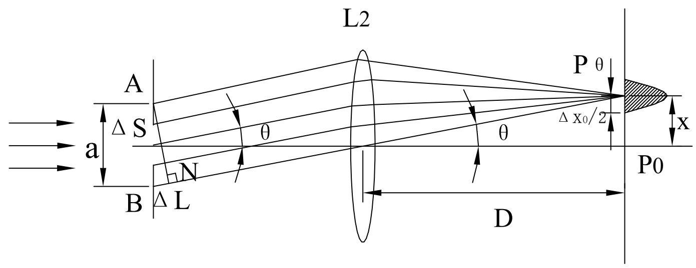

Figure 1 Schematic of Fraunhofer diffraction from single slit

Fraunhofer diffraction is the diffraction of parallel light rays, the so-called far-field diffraction, which can be treated and explained by using the Huygens' Principle. As a plane wave is incident on a long, narrow slit, secondary waves are emitted. These secondary waves are spherical, as if there were an infinite number of point sources across the aperture. From a particular observation point, each source has a different optical path that introduces a phase relationship between the waves emitted from the aperture. The total intensity of the waves emitted from the aperture bears a simple relationship with regard to the angle of diffraction, as described in many textbooks. In the observation plane, we may write:

$$I = I_0 \frac{\sin^2(u)}{u^2} \tag{1}$$

where $u = \frac{2\pi}{\lambda}\frac{a}{2}\sin\theta$, a: the slit-width, λ: the light wavelength, and θ: the angle of diffraction.

When u=0, namely, θ=0, maximum diffraction occurs, so the relative intensity is I/I_0=1;

When $u=n\pi$, namely, $a\sin\theta=n\lambda$, where n is an integer, minimum diffraction occurs (I=0).

Let $\frac{d}{du}(\frac{\sin u}{u}) = 0$, we get the conditions of secondary maxima:

$$u = \tan(u) \tag{2}$$

By calculating, we have the following table:

u	$\sin(\theta)$	I/I_0
$\pm1.43\pi$	$\pm1.43\lambda/a$	0.047
$\pm2.46\pi$	$\pm2.46\lambda/a$	0.017
$\pm3.47\pi$	$\pm3.47\lambda/a$	0.008
…	…	…

From the above table, the intensity of Fraunhofer diffraction at a single slit is illustrated in the following figure.

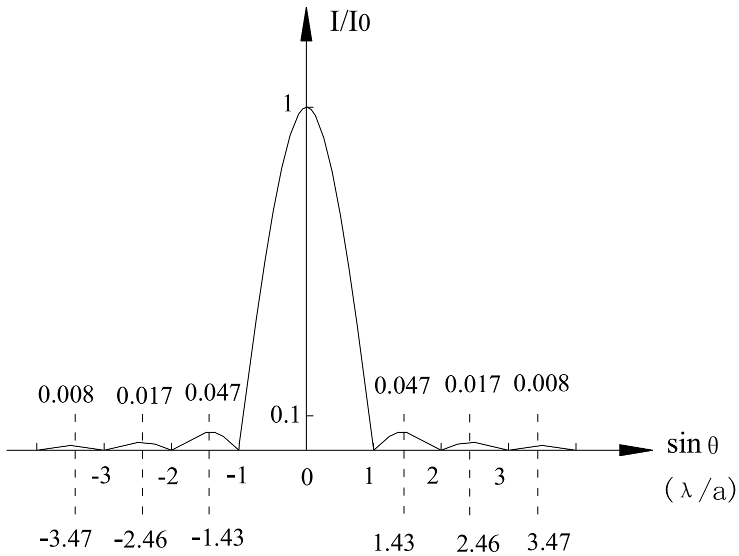

Figure 2 Intensity distribution of Fraunhofer diffraction at single slit

The divergence angle of a collimated diode laser beam is very small, so the laser beam can be considered as a parallel beam. A laser can be used as the source for Fraunhofer diffraction as shown in Figure 3. If the distance between the detector and the slit is far enough so that the difference between AP_0 and OP_0 is far less than the laser wavelength, λ, the condition of Fraunhofer diffraction is satisfied. Let's prove it.

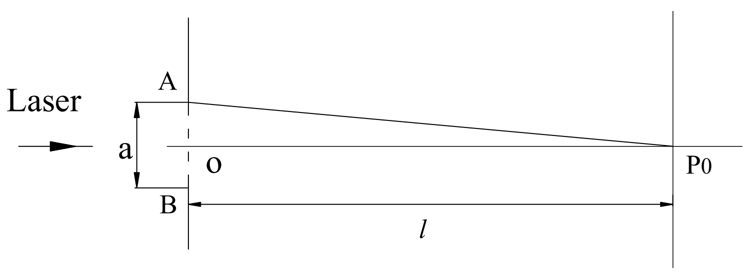

Figure 3 Schematic of far-field Fraunhofer diffraction

If the slit-width, a, is considered far less than the observation distance from the slit, l, then

$$AP_0\text{-}OP_0=\sqrt{l^2+\frac{a^2}{4}}-l\approx l\left(1+\frac{a^2}{8l^2}\right)-l=\frac{a^2}{8l} \tag{3}$$

In the experiment, for example if $l = 0.8$ m, $a = 0.1$ mm, then

$\frac{a^2}{8l}$=15.6 nm, which is far less than λ (632.8 nm). Hence, the far-field condition for Fraunhofer diffraction is satisfied.

4.2 Fraunhofer Diffraction of Multi-Slit

When the single slit is replaced by a 'multi-slit' that has N evenly-spaced slits with a width of a for each slit and a slit-to-slit separation of d, the resulted intensity of diffraction is the intensity distribution from N coherent sources multiplied by the diffraction pattern from a single slit. The interference pattern is determined by the phase angle $\beta=(\pi\times d\times\sin\theta)/\lambda$, where λ is the wavelength of light, and θ the angle of diffraction or angle of observation. Because the diffraction pattern is determined by $u=(2\pi\times a\times\sin\theta)/2\lambda$, the net result is

$$I = I_0\left(\frac{\sin u}{u}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 \tag{4}$$

where $(\sin u)/u$ or $(\sin u)^2/u^2$ is a diffraction function and $(\sin N\beta)/(\sin\beta)$ or $(\sin N\beta)^2/(\sin\beta)^2$ is an interference function.

Based on equation (4), the following picture shows the normalized interference function with different N values,

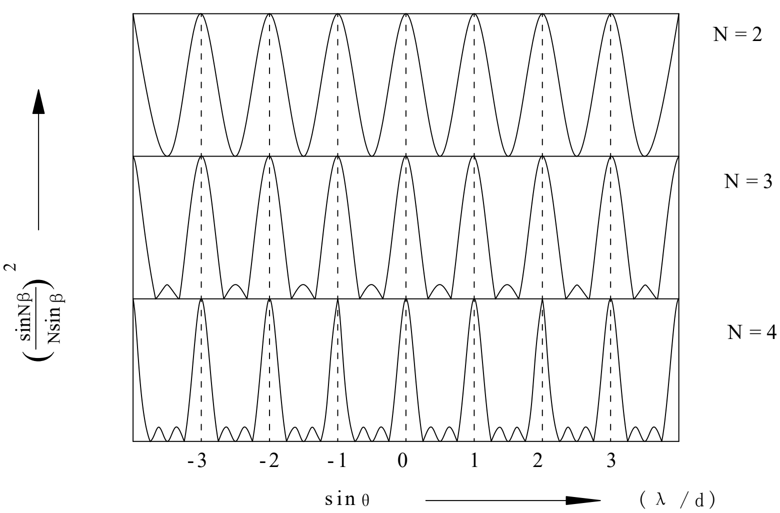

Figure 4 Normalized interference intensity distribution of N coherent sources

1) *__Peak values, positions, and numbers of the primary maxima__*

When $\beta = k\pi$, namely $\sin\theta = k\frac{\lambda}{d}$ $(k = 0, \pm 1, \pm 2, \pm 3, \cdots)$, then

$$I = I_0\left(\frac{\sin u}{u}\right)^2\left(\frac{\sin N\beta}{\sin\beta}\right)^2 = N^2 I_0\left(\frac{\sin u}{u}\right)^2 \tag{5}$$

Obviously, the principal maximum diffraction intensity from a multi-slit with N evenly-spaced slits is N^2 times the diffraction intensity at a single slit.

Considering $|\sin\theta| \le 1$, so $|k| \le d/\lambda$

Let $\lambda=0.4d$, then k can only take $0, \pm 1, \pm 2$.
If $\lambda \ge d$, then $k=0$, it means there is only zero order.

2) <u>Positions of zero intensity points, numbers of the secondary maxima</u>

Between these principal maxima there are weak secondary maxima. Let $\sin N\beta = 0$ and $\sin\beta \ne 0$, we can get these positions of zero intensity points.

$$\beta = (k + \frac{m}{N})\pi \text{, namely } \sin\theta = (k + \frac{m}{N})\frac{\lambda}{d} \quad (6)$$

where $k = 0, \pm 1, \pm 2, \cdots$; $m = 1, 2, \cdots, N-1$.

Because there are (N–1) zero points between two principal maxima, and there is a secondary maximum between the consecutive zero points, the number of secondary maxima is N–2. The following figure shows the influence of the diffraction function (N = 5, d = 3a).

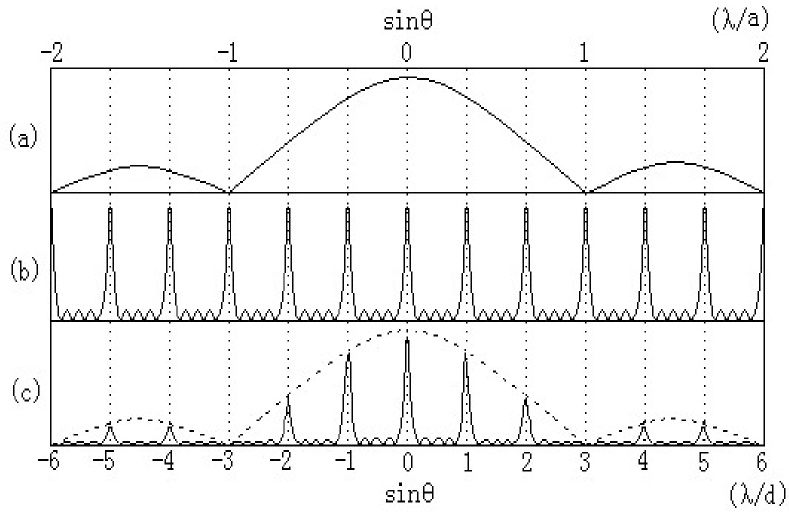

Figure 5 Diffraction intensity (c) of 5-slit (N = 5, d = 3a)

It shows that the interference function (b) is modulated by the diffraction function (a), yielding the net result shown in (c). For d=3a, the third order (k=3) principal maximum of the interference function coincides with the first zero position of single slit diffraction. This is the so-called missing order phenomenon.

4.3 Babinet's Principle and Fraunhofer Diffraction of Thin Wire

Babinet's principle states that at any location, the superposition of the two diffraction wave fields from two diffracting screens with complementary patterns is equal to the wave field of the light propagating freely to that location. Babinet's principle can be illustrated in Figure 6. If the diffraction wave field of a diffracting screen at location P is u_1 (p) while the diffraction wave field of its complement at location P is u_2 (p), and the wave field of light propagating freely to the same location is u_0 (p), then we get the following equation based on Babinet's principle:

$$\tilde{u}_1(p) + \tilde{u}_2(p) = \tilde{u}_0(p) \quad (7)$$

Figure 6 Schematic of Babinet's principle

Babinet's principle can be applied to any point in the far field or the near field so long as the illuminating condition is kept identical for the two complementary diffracting screens. Based on Babinet's principle, if the diffraction wave field of a diffracting screen is known, then the diffraction wave field of its complement can be derived from the wave field of light in obstacle-free space. For example, if the diffraction wave field of an aperture is known, then the diffraction wave field of a disk of the same diameter can be obtained; similarly, if the diffraction wave field of a single slit is known, then the diffraction wave field of a wire whose cross sectional diameter is equal to the slit width can be obtained.

Babinet's principle is very effective when dealing with Fraunhofer diffraction of various objects. Except geometrical image locations, the wave field at any other locations in obstacle-free space is zero thus $u_1\ (p)=u_2\ (p)$ leading to $I_1\ (p)=I_2\ (p)$. At geometrical image locations, the diffraction intensities of the two complementary screens should be maximum although the intensity values may be different. It should be noted that Babinet's principle states the relationship among three wave fields rather than that among three intensities. Due to phase delay effect, if the diffraction intensity of a diffracting screen is maximum at a certain location, then the diffraction intensity of its complement at the same location is not necessarily minimum.