2. Theory

A. Derivation of parabolic formula of rotating liquid

In the rotating reference system, this is a static equilibrium issue. The liquid volume element at the liquid surface is affected by external forces as well as adjacent liquid. Since the liquid does not move tangentially on the liquid surface, the total external force on the liquid surface should be perpendicular to the liquid surface in order to balance with the internal force of the liquid, as shown in Figure 1.

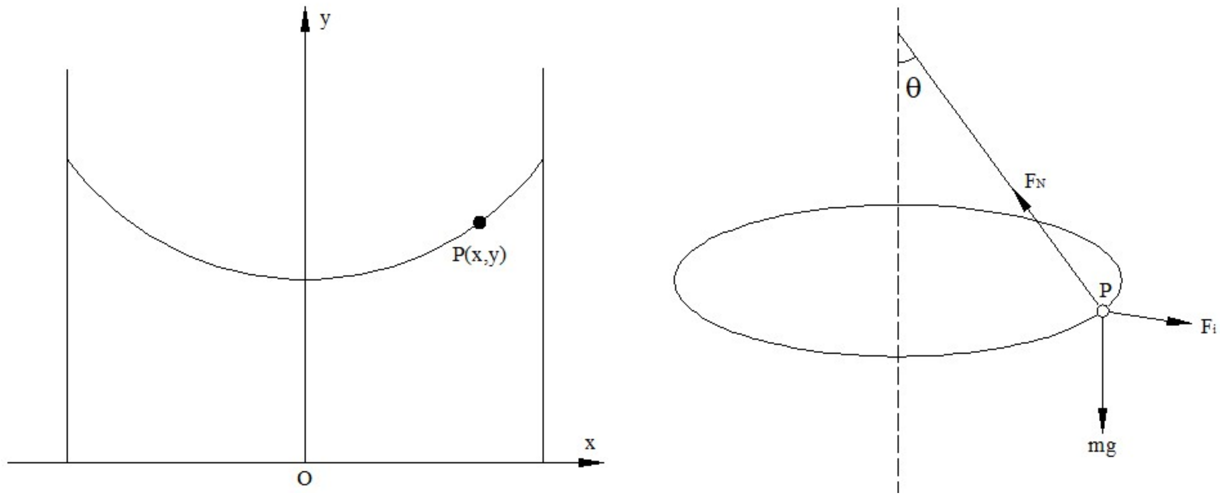

Figure 1 Schematic of force states on liquid surface

When the container does not rotate, the external force applied to the volume element of liquid surface is only the gravity, which is vertically downward, so the liquid surface is flat in horizontal. When the cylindrical container rotates, in addition to the gravity, it also receives inertial centrifugal force, and the farther away from the axis of rotation, the greater the inertial centrifugal force. The combined force deviates from the vertical direction, and the deviation becomes larger where closer to the container edge. The liquid surface should be perpendicular to the combined force, so it takes the concave shape of a paraboloid.

For the quantitative calculation, the rotating reference system of the cylindrical container is selected, which is a rotating non-inertial reference system. The liquid is stationary relative to the reference system. Choose a small piece of liquid P at point (x,y), its force situation is shown in Figure 1. F_i is the inertial centrifugal force radially outward, m_g is the gravity, F_N is the combined force which is received from the adjacent liquid. From the symmetry, it can be seen that F_N must be perpendicular to the liquid surface. In the coordinate system O-xyz (Oy axis is along the symmetry axis of the container, Oxy plane is on the symmetry plane), for the small piece of liquid P, we have the following equation:

$$F_N\cos\theta - mg = 0, \tag{1}$$
$$F_N\sin\theta - F_i = 0, \tag{2}$$
$$F_i = m\omega^2 x, \tag{3}$$

where ω is the rotating angular velocity of the cylindrical container.

From formulas (1), (2) and (3), we can get:

$$\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g}. \tag{4}$$

The solution of equation (4) is

$$y = \frac{\omega^2}{2g}x^2 + y_0, \tag{5}$$

where y_0 is the y value at $x = 0$. This is a parabolic equation, and it can be seen that the liquid surface is a rotating paraboloid.

B. Measurement of gravity acceleration *g* using rotating liquid

In the experimental system, when a cylindrical container of radius *R* containing liquid rotates steadily at an angular velocity ω around the symmetric axis of the cylinder, the surface of the liquid forms a paraboloid, as shown in Figure 2.

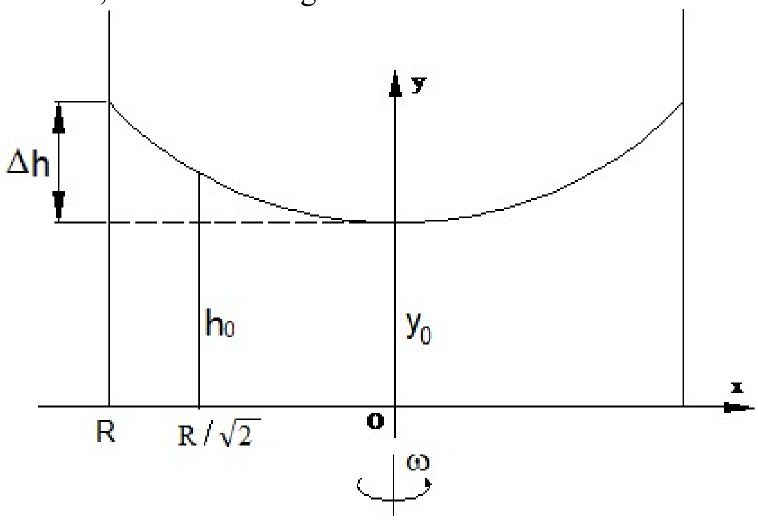

Figure 2 Axial sectional view of rotating liquid

Let the height of the liquid surface be h_0 when the liquid is not rotating, and the volume of the liquid is:

$$V = \pi R^2 h_0. \tag{6}$$

The volume of the liquid remains unchanged before and after the rotation. During the rotation it can be expressed as:

$$V = \int_0^R y(2\pi x)dx = 2\pi\int_0^R \left(\frac{\omega^2 x^2}{2g} + y_0\right)x dx, \tag{7}$$

From formulas (6) and (7), we get:

$$y_0 = h_0 - \frac{\omega^2 R^2}{4g}. \tag{8}$$

From formulas (5) and (8), when $x = x_0 = \frac{R}{\sqrt{2}}$, we have $y(x_0) = h_0$, that is, the height of the liquid level at point x_0 is a constant.

There are two methods for measuring the acceleration of gravity as follows.

(1) Measure the gravity acceleration using the height difference between the highest and lowest points of the rotating liquid surface

As shown in Fig. 2, suppose the height difference between the highest and lowest points of the rotating liquid surface is Δh, and the point $(R, \ y_0 + \Delta h)$ (the intersection of the liquid surface and the container wall) is on the paraboloid of formula (5), we have:

$$y_0 + \Delta h = \frac{\omega^2 R^2}{2g} + y_0. \tag{9}$$

The relationship between the height difference and the rotation speed can be obtained as:

$$\Delta h = \frac{\omega^2 R^2}{2g}. \tag{10}$$

Since $\omega = \frac{2\pi}{T}$, we then have:

$$g = \frac{2\pi^2 R^2}{T^2 \Delta h}. \tag{11}$$

According to formula (11), the acceleration of gravity can be calculated as a function of the rotation speed of the container and the rising height of the liquid edge.

The disadvantage of this method is that usually the height difference between the highest and lowest points of the liquid surface cannot be measured accurately.

(2) Measure the gravity acceleration by slope measurement method using a parallel laser beam

As shown in Figure 3, the laser beam is incident parallel to the rotation axis through the plane where the observation screen is located and hits the liquid surface A at $x_0 = \frac{R}{\sqrt{2}}$. The incident light point and reflected light point on the plane are respectively B and C. Assume the angle between the tangent of liquid surface and the x direction is θ at point A, then $\angle BAC = 2\theta$. By measuring the height H (from the observation screen to the bottom of the container), the height h_0 (when the container does not rotating), and the distance d (between the two points B and C), we have $\tan 2\theta = \frac{d}{H - h_0}$, from which we obtain the value of $\tan(\theta)$.

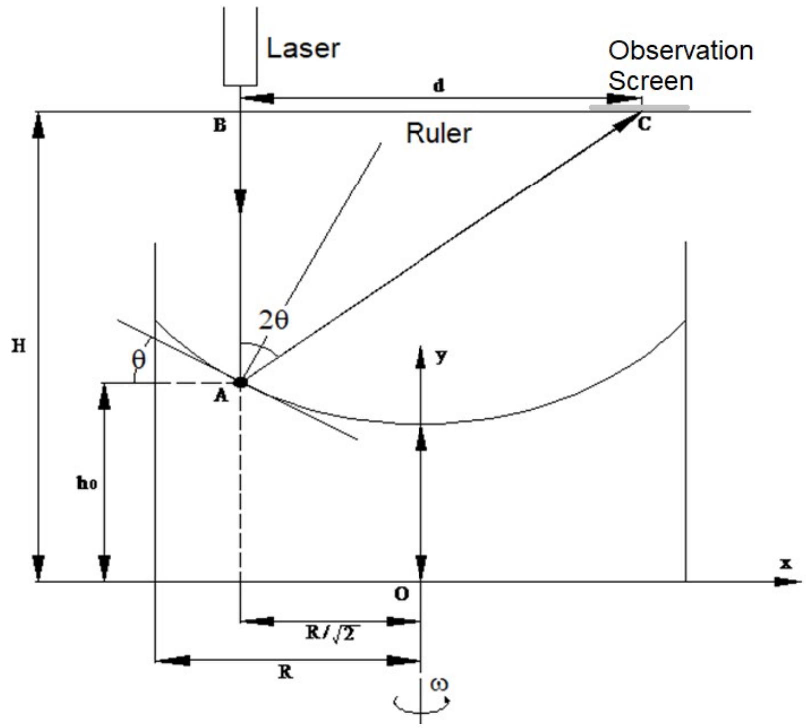

Figure 3 Schematic of measuring gravity acceleration using a parallel laser beam

Since $\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g}$ and $\omega = \frac{2\pi}{T}$, at $x_0 = \frac{R}{\sqrt{2}}$, we have:

$$\tan\theta = \frac{\omega^2 R}{\sqrt{2}g} = \frac{2\sqrt{2}\pi^2 R}{gT^2}. \tag{12}$$

According to the formula (12), plot $\tan\theta - \frac{1}{T^2}$ relationship graph, and use the least square method to do a linear fitting to obtain the slope *k*, *g* can be acquired from the slope *k*:

$$g = \frac{2\sqrt{2}\pi^2 R}{k}. \tag{13}$$

C. Verify relationship of focal length *f* and rotation period *T* according to parabolic equation

The paraboloid formed by the rotating liquid surface can be considered as a concave mirror of partial reflection, which conforms to the rules of an optical imaging system. If the light is incident parallel to the symmetry axis of the curved surface, the reflected light will converge to the focal point of the parabolic surface, whose position can be found on the rotation axis.

According to the parabolic equation (5), the focal length of the parabola is $f = \frac{g}{2\omega^2}$. With $\omega = \frac{2\pi}{T}$, we get:

$$f = \frac{g}{8\pi^2} T^2. \tag{14}$$

In the experiment, assuming $f = aT^b$, then $\ln f = b\ln T + \ln a$. Change the rotation period *T* to obtain multiple sets of *f* and *T* values, make the $\ln f - \ln T$ relationship plot, and then use the least square method to do linear fitting to obtain values of *a* and *b*. Comparing with equation (14), we can verify whether the rotating liquid surface conforms to the parabolic shape.

D. Study on imaging rules of the concave mirror of rotating liquid

The paraxial surface of the rotating liquid can also be approximated as a concave mirror of partial reflection with curvature radius *r*, object distance *S*, and image distance *S*', then, according to the imaging formula $\frac{1}{S}+\frac{1}{S'}=\frac{1}{f}$, (note $f = r/2$), the imaging states of a concave mirror can be summarized in Table 1 below.

Table 1 Concave mirror imaging rules

Object	Image	Image size	Real/Virtual	Image Direction
$S=\infty$	$S'=f$	Focused to a spot	Real	Reversed
$S>r$	$f<S'<r$	Reduced	Real	Reversed
$S=r$	$S'=r$	Equal	Real	Reversed
$r>S>f$	$S'>r$	Enlarged	Real	Reversed
$S=f$	$S'\to\infty$	N/A	N/A	N/A
$0<S<f$	$S'<0$	Enlarged	Virtual	Erected
$S<0$	$S'>0$	Reduced	Real	Erected

Note: For S <0, the image is converged behind the concave mirror as a real image (due to partial reflection).

At a certain rotation period *T*, if setting object distance to $S=r$, a real image of equal size will be formed at *r*, so we obtain the measured value of focal length $f'=\frac{r}{2}$. According to equation (14), the theoretical value of focal length is $f=\frac{g}{8\pi^2}T^2$. Comparing the measured value with the theoretical value, we can verify the imaging rules of the concave mirror of rotating liquid.