2. Theory

All liquids are viscous, which means there exists a friction between fluid layers when liquid flows. This friction occurs inside the liquid so it is called the internal frictional force. Internal friction is a result of the forces between molecules under molecular thermal motion. The strength of internal friction is related to the area of the fluid layer, the velocity gradient dv/dx between two layers, and the characteristics of the fluid itself.

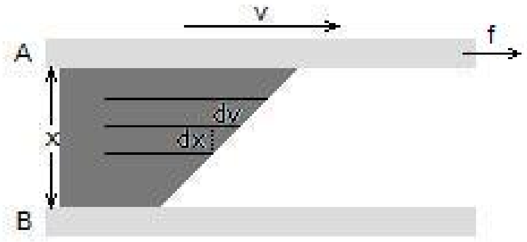

Figure 1 Schmetic of internal friction coefficient and velocity distribution

In Figure 1, if we assume a fluid is filled between two parallel plates A and B with equal area S separated by distance x. Plate B remains stationary while a constant force f is applied to plate A in the tangential direction of the surface. Since there exists friction between the liquid layer attached to the surface of plate A and the layer between the two plates, plate A undertakes a uniform motion at speed of v from acceleration movement. Velocity distribution of each layer is shown in Figure 1. As a result, force f is equal to internal friction f'. We have:

$$f = \eta \frac{dv}{dx} S \tag{1}$$

where η is the liquid viscosity coefficient. So η is:

$$\eta = \frac{f/S}{dv/dx} \tag{2}$$

The viscosity coefficient of a fluid is related to fluid properties and temperature, in unit of Pa·s (1 Pa·s =1 N· s/m^2). In this experiment, the viscosity coefficient of a fluid is measured using the capillary method. If we assume the actual liquid takes a steady flow in a horizontal tube of radius R and length L, the radius of the liquid column is r ($r<R$), and the pressure difference on the two ends of the liquid column is P_1-P_2, the force pushing the liquid column to flow is:

$$F_1 = (P_1 - P_2)\pi r^2 \tag{3}$$

The viscous resistance of the liquid column is:

$$F_2 = -\eta \frac{dv}{dr} 2\pi r L \tag{4}$$

Since the fluid is in steady flow, we have:

$$F_1 = F_2 \rightarrow (P_1 - P_2)\pi r^2 = -2\pi r L\, \eta \frac{dv}{dr} \rightarrow -\frac{dv}{dr} = \frac{P_1 - P_2}{2\eta L} r$$

By taking an integral operation to above equation, we get:

$$v = \frac{P_1 - P_2}{4\eta L}(R^2 - r^2) \tag{5}$$

Within time *t*, the liquid volume flowing through an arbitrary cross section in the tube is:

$$V = \int_0^R 2\pi r v t\, dr = \frac{\pi R^4 (P_1 - P_2)}{8\eta L} t \tag{6}$$

Equation (6) is the Poiseuille formula, which can be rewritten as:

$$\eta = \frac{\pi R^4 (P_1 - P_2)}{8VL} t \tag{7}$$

Using the above formula, one can calculate the viscosity coefficient of a liquid.

The structure of an Ostwald viscometer is shown in Figure 2. A U-shaped glass tube is placed vertically. A certain volume of liquid is injected from *a*. Liquid surface is approximately in the middle of ball *b*. During measurement, one needs to inhale liquid into ball *c* above mark line *m*, and let the liquid segment flow freely down in the capillary. When the surface passes through mark line *m*, start timing; when liquid surface passes through mark line *n*, stop timing.

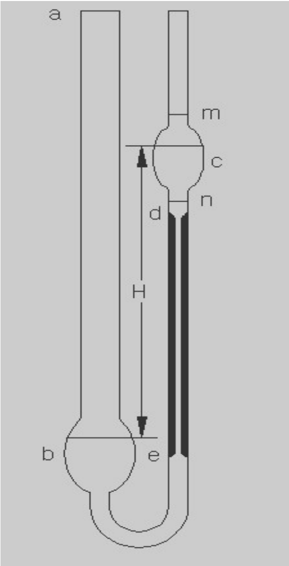

Figure 2 Schematic of Ostwald viscometer

Volume V inside the tube between mark lines m and n is the liquid volume flowing through the capillary in time t. The force pushing the liquid to flow is determined by height difference H between the two surfaces in the two sides of the U-tube, as

$$P_1 - P_2 = \rho g H \tag{8}$$

By substituting (8) to (7), we have:

$$\eta = \frac{\pi R^4 g H}{8VL} \rho t \tag{9}$$

In actual measurement, however, it is difficult to accurately measure radius R, length L, and volume V between mark lines m and n of the capillary. Moreover, height difference H between two liquid surfaces is also variable during liquid flow. To overcome these issues, a comparative method is introduced (a liquid is used as the calibration sample to measure an unknown liquid) as described below.

By using the same Ostwald viscometer to measure two liquids, we have:

$$\eta_1 = \frac{\pi R^4 g H}{8VL} \rho_1 t_1 \tag{10}$$

$$\eta_2 = \frac{\pi R^4 g H}{8VL} \rho_2 t_2 \tag{11}$$

Since R, V and L are all fixed values, height difference H at the start and end of measurement is identical if the volume of the two liquids is also identical. So, we get:

$$\frac{\eta_1}{\eta_2} = \frac{\rho_1 t_1}{\rho_2 t_2} \tag{12}$$

Or

$$\eta_2 = \eta_1 \frac{\rho_1 t_1}{\rho_2 t_2} \tag{13}$$

If η_1, ρ_1 and ρ_2 are known, by measuring t_1 and t_2, η_2 can be calculated.

In addition, an Ostwald viscometer can also be used to measure the surface tension coefficient of a liquid. As the required quantity of sample is tiny, using an Ostwald viscometer to measure the coefficient of viscosity and surface tension coefficient of a liquid is convenient in practice.

As shown in Figure 2, one side of an Ostwald viscometer is a glass tube with a slightly larger diameter while the other side is a very narrow capillary, which is equivalent to the case where a capillary is inserted into liquid. When the liquid is stationary, due to capillary action, the liquid will rise by a certain height $\mathrm{D}h$ along the capillary.

At a certain temperature, if a standard liquid with known surface tension coefficient a_1 and density ρ_1, falls freely from ball c to reach at a stationary state with equal pressure at both sides, the liquid heights in right capillary and left tube are h_{01} and h_1, respectively; we have;

$$a_1 = (h_{01} - h_1) r \rho_1 g / 2 = Dh_1 r \rho_1 g / 2 \tag{14}$$

where $Dh_1 = h_{01} - h_1$, is the height difference between right capillary and left tube. Similarly, for another liquid with the same volume and temperature, we have;

$$\alpha_2 = \mathrm{D}h_2 r \rho_2 g / 2 \tag{15}$$

From (14) and (15), we get:

$$\alpha_2 = \frac{\mathrm{D}h_2 \rho_2}{\mathrm{D}h_1 \rho_1} \alpha_1 \tag{16}$$

Therefore, a_2 can be derived from Eq. (16) by measuring $\mathrm{D}h_1$ and $\mathrm{D}h_2$.