2. Structure and Specifications

1) Apparatus structure

As shown in Figure 1, this apparatus consists of a mechanical resonant vibration unit and an electric control unit. A copper circular balance wheel is installed on a rack. Angle scales are marked on the outer ring of the balance wheel.

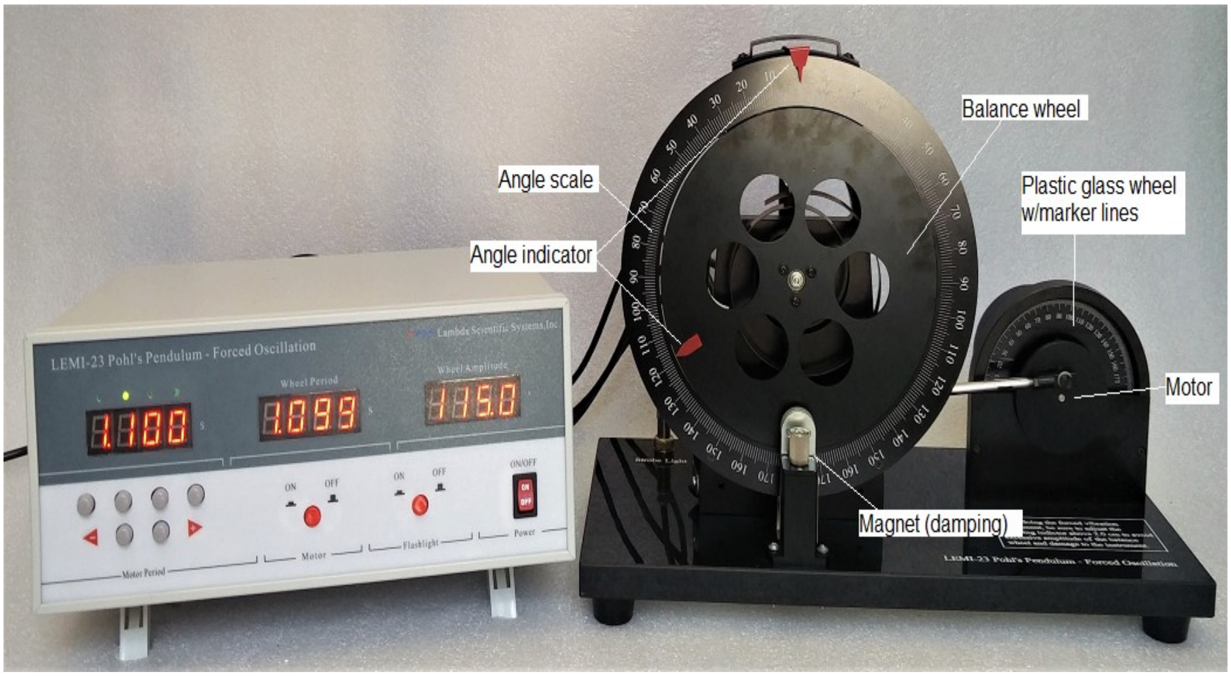

Figure 1 Photo of the forced vibration and resonance apparatus

One end of the spring is connected to the shaft of the balance wheel and the other end is fixed on the rack post. Under the impact of the spring force, the balance wheel swings freely around its shaft. There is a pair of permanent magnets in the lower part of the rack and the balance wheel is located in the magnet gap. Due to electromagnetic induction, when the balance wheel cuts magnetic lines, it is subject to an electromagnetic damping force. The damping magnitude is changed by changing the location of the magnets. In order to make the balance wheel to do forced vibration, an eccentric wheel is mounted on the motor shaft through a connecting-rod mechanism to drive the balance wheel. A plastic glass wheel with engraved marker lines is mounted on the motor shaft and rotates with the motor. The phase difference φ can be read out on the disk. The rotation speed of the motor can be precisely adjusted by the control unit.

When forced vibration occurs, the phase difference between the balance wheel and the external torque is measured using an LED flashlight. The flashlight is controlled by the reflective photoelectric gate of the balance wheel. Whenever the black line on the inner side of the balance wheel passes through the reflective photoelectric gate, a flash is triggered. Under stable conditions, the plexiglass pointer can be seen under the illumination of the flash as if it has been "stopped" at a certain scale. This phenomenon is called stroboscopic phenomenon, so this value can be easily read directly, and the error is not greater than 2°.

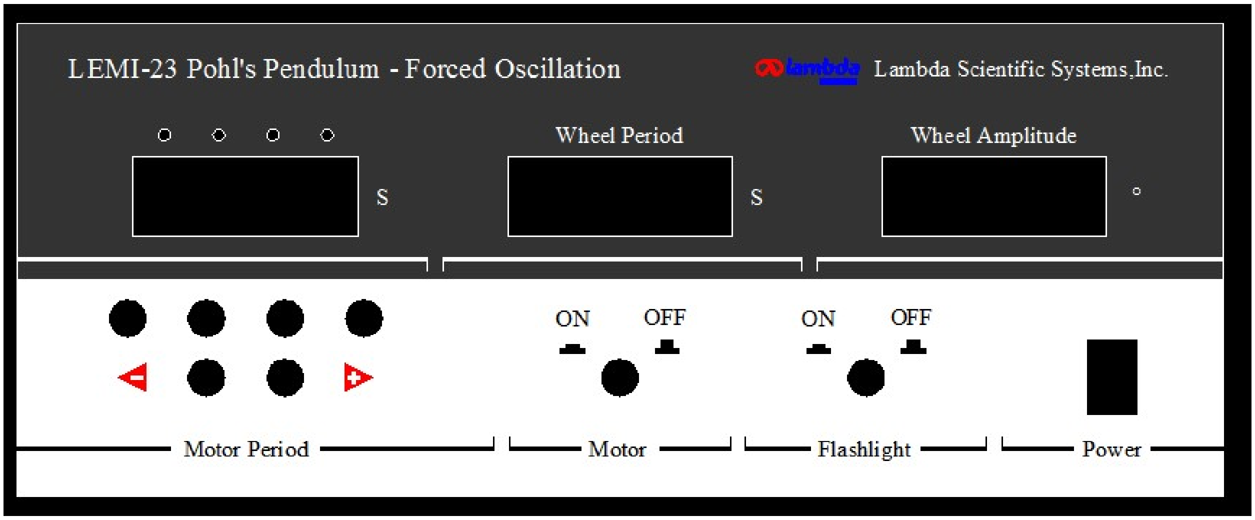

Figure 2 Schematic drawing of front panel

The schematic drawing of the front panel of the electric control unit is shown in Figure 2. The middle digital meter shows the period of the copper balance wheel, and the digital meter on the left shows the rotation period of the motor. When the system is stable, the two values should be the same, and the error should not exceed 2×10^{-3}s. The meter on the right displays the amplitude of the balance wheel with error less than 1°.

There are six buttons under the motor period meter. The buttons in the upper row are used to select the bits of the motor period. The two buttons in the next row are used to increase or decrease the value of the corresponding bits of the digital meter with the left button for decrease and the right one for increase.

The magnitude of damping can be changed by adjusting the up and down position of the magnets. The more the magnetic force lines pass through the balance wheel, the greater the damping.

The flashlight switch is used to control the flash. When the button is pressed, the flash is in working state. The motor switch is used to control whether the motor rotates or not. When the button is pressed, the motor works. When measuring the damping coefficient and the natural frequency of the balance wheel, the electric motor must be turned off.

2) Specifications

Description	**Specifications**
Spring stiffness coefficient K	Variation of free vibration period: <1%
Time measurement	Accuracy: 0.001 s; error of period: 0.2%; 4-digit display
System damping	Amplitude attenuation < 2% without electromagnetic damping
Amplitude measurement	Error: ± 1 °
Motor rotational speed	Range: 15 ~ 50 r/min; period adjustable: 0.2 ~ 4 s
Phase difference measurement	Error < 2 ° when phase difference between 40 ~ 140 °

3. Theory

1) Forced vibration

Vibration of an object under the continuous action of a periodic external force is called forced vibration.

When the balance wheel is subjected to periodic forced torque $M = M_0 \cos\omega t$ and moves in a medium with air damping and electromagnetic damping (the damping torque is $-b\frac{d\theta}{dt}$), its forced vibration equation is:

$$J\frac{d^2\theta}{dt^2} = -k\theta - b\frac{d\theta}{dt} + M_0 \cos\omega t\ , \tag{1}$$

Where J is the rotational inertia of the balance wheel, $-k\theta$ is the elastic moment, M_0 is the amplitude of the forcing moment, and ω is the circular frequency of the driving force. Let $\omega_0^2 = k/J$, $2\beta = b/J$, $m = M_0/J$, then the above formula becomes:

$$\frac{d^2\theta}{dt^2} + 2\beta\frac{d\theta}{dt} + \omega_0{}^2\theta = m\cos\omega t\ . \tag{2}$$

When $m\cos\omega t = 0$, equation (2) is the damped vibration equation.

If β is also 0, then equation (2) is a simple harmonic motion equation, and the natural frequency of the system is ω_0.

The general solution of equation (2) is a special solution plus the general solution of the homogeneous equation,

$$\theta = \theta_1 e^{-\beta t}\cos(\omega_f t + \alpha)\,.$$

It represents an amplitude-reduced vibration, reflecting the transient behavior of forced vibration, which decay to disappear after a short time.

Assume a special solution of this equation is $\theta = \theta_2\cos(\omega t + \varphi)$, then:

$$\begin{cases} \dfrac{d\theta}{dt} = -\theta_2\omega\sin(\omega t + \varphi) \\ \dfrac{d^2\theta}{dt^2} = -\theta_2\omega^2\cos(\omega t + \varphi) \end{cases}$$

Substitute into the equation and sort out:

$$\begin{cases} \theta_2(\omega_0{}^2 - \omega^2)\cos\varphi - 2\beta\theta_2\omega\sin\varphi = m \\ \theta_2(\omega_0{}^2 - \omega^2)\sin\varphi - 2\beta\theta_2\omega\cos\varphi = 0 \end{cases}$$

From the above formula, we get:

$$tg\varphi = \frac{-2\beta\omega}{\omega_0{}^2 - \omega^2}$$

And we further get: $$\begin{cases} \sin\varphi = \dfrac{-2\beta\omega}{\sqrt{(\omega_0^{\ 2}-\omega^2)+4\beta^2\omega^2}} \\ \cos\varphi = \dfrac{\omega_0^{\ 2}-\omega^2}{\sqrt{(\omega_0^{\ 2}-\omega^2)+4\beta^2\omega^2}} \end{cases}$$

Substitute into the formula (2), we get:

$$\theta_2 = \frac{m}{\sqrt{(\omega_0^{\ 2}-\omega^2)+4\beta^2\omega^2}}\ . \tag{3}$$

It can be seen from equation (3) that the amplitude θ_2 is related to the amplitude $M_0(M_0 = mJ)$ of the forcing torque, the circular frequency ω, the natural frequency ω_0 of the system and the damping coefficient β, but not related to the initial state.

2) Displacement resonance

By extreme conditions: $\frac{\partial}{\partial\omega}[(\omega_0^{\ 2}-\omega^2)^2+4\beta^2\omega^2]=0$

It can be derived that when the circular frequency of the driving force $\omega{=}\sqrt{\omega_0^{\ 2}-2\beta^2}$, resonance occurs. θ has a maximum value, which is called displacement resonance. The conditions of displacement resonance are:

$$\begin{cases} \omega_r{=}\sqrt{\omega_0^{\ 2}-2\beta^2} < \omega_0 & (4) \\ \theta_r = \dfrac{m}{2\beta\sqrt{\omega_0^{\ 2}-\beta^2}} & (5) \end{cases}$$

Equations (4) and (5) show that the smaller the damping coefficient β, the closer the resonance circle frequency ω_r to the natural frequency ω_0 of the system, the greater the amplitude θ_r.

When displacement resonance occurs, the phase difference between displacement and forcing force is:

$$|\varphi_r| = tg^{-1}\frac{2\beta\omega_r}{\omega_0^2-\omega_r^2} = tg^{-1}\frac{\sqrt{\omega_0^2-2\beta^2}}{\beta}$$

When the damping is very small, $tg\varphi_r \to -\infty, \varphi_r \to -\frac{\pi}{2}$

3) Speed resonance and acceleration resonance

When the system is forced to vibrate, the velocity amplitude is $V_m = \omega\theta_2$. As can be seen from equation (3), when $\omega{=}\ \omega_0$, θ has a maximum value (speed resonance condition).

Because the speed vibration phase leads the displacement vibration phase by $\pi/2$, the speed vibration and the forcing force are in the same phase. Therefore, the forcing force always does positive work on the system, and the system is continuously replenished with energy, so resonance occurs.

When the system is forced to vibrate, its acceleration amplitude is $a_m = \omega^2 \theta_2$. Since $da_m / d\omega = 0$, we know:

When $\omega = \frac{\omega_0^{\ 2}}{\sqrt{\omega_0^{\ 2} - 2\beta^2}} > \omega_0$, a_m has a maximum value (acceleration resonance condition).

The three resonances have large amplitudes, and among them, the displacement resonance has the largest amplitude.

4) Amplitude-frequency characteristics and phase-frequency characteristics

The amplitude-frequency characteristic and phase-frequency characteristic are shown in Figure 3 and Figure 4 respectively.

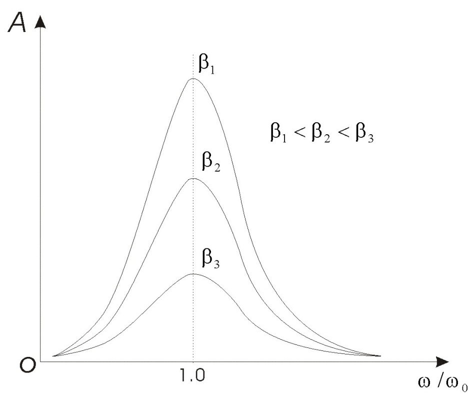

Figure3 Amplitude-frequency characteristic

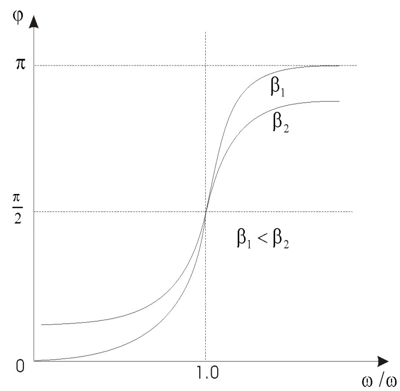

Figure 4 Phase-frequency characteristic