2. Theory

1) Relationship between period and swing angle

If neglecting air resistance and buoyancy, from the standpoint of energy conservation of a swinging pendulum, the summation of the kinetic energy and potential energy of the pendulum ball of mass *m* at angle θ will be a constant, namely:

$$\frac{1}{2} mL^2 \left(\frac{d\theta}{dt} \right)^2 + mgL\,(1 - \cos\,\theta) = E_0 \tag{1}$$

where L is the length of the pendulum string, θ is angle, g is gravitational acceleration, t is time, and E_0 is the total mechanical energy of the ball.

Assuming the ball is released at swing angle θ_m, we have:

$$E_0 = mgL(1 - \cos\theta_m) \tag{2}$$

By substituting (2) into (1) and taking some mathematical operations, we get:

$$\frac{\sqrt{2}}{4} T = \sqrt{\frac{L}{g}} \int_0^{\theta_m} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_m}} \tag{3}$$

where T is the period of swing of the pendulum.

Let $k = \sin(\theta_m / 2)$, by making $\sin(\theta / 2) = k \sin\varphi$, we get:

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2\varphi}} \tag{4}$$

This is an elliptic integral, by approximation we get:

$$T = 2\pi\sqrt{\frac{L}{g}} \left[1 + \frac{1}{4}\sin^2(\frac{\theta_m}{2}) + \Lambda \right] \tag{5}$$

Under the traditional method of manual timing, the measurement error of a single period could be up to 0.1 ~ 0.2 second, while a measurement of multiple periods will experience a decay of swing angles due to air damping (i.e. θ_m is reduced with an increase in swing times). Therefore, only the first-order approximation can be taken into account in (5) by ignoring the second-order term $\frac{1}{4}\sin^2(\frac{\theta_m}{2})$ and the following higher-order terms. On the other hand, to accurately measure the period of a pendulum, the influence of swing angle on period must be considered so that the second-order approximation should be employed in Equation (5).

In this experiment, we will acquire the time of two periods (i.e. $2T$) for different swing angles θ_m, plot the graph of $2T \sim \sin^2(\frac{\theta_m}{2})$, and extrapolate the data to derive the intercept of the graph for θ_m=0 to get period T. Finally, gravitational acceleration g can also be obtained.

2) Relationship between period and string length

If one end of a non-stretchable string without mass is fixed at a point with the other end of the string hung by a point object of mass m, a simple pendulum is built. When the swing angle θ_m is small (e.g. < 3°), there is an approximate relationship between period T of the pendulum and length L of the string, as

$$T = 2\pi\sqrt{\frac{L}{g}} \text{ or } T^2 = 4\pi^2\frac{L}{g} \tag{6}$$

In reality, a simple pendulum does not exist, because the string has certain mass, and the small ball has a certain radius (r) and is not a point object. Therefore, only when the mass of the ball is much larger than that of the string and the radius of the ball is much smaller than the string length can Equation (6) be valid. Under such case, the length of pendulum L is the distance between the fixing point and the center of ball, i.e. $L=L_1+r$, where L_1 is the length of the string.

If the pendulum length L is fixed, through measurement of period T, g can be derived from (6). Alternatively, one can successively change pendulum length L, measure the corresponding period T, and plot $T^2 \sim L$ graph. If the graph is a straight line, it means T^2 is proportional to L. By selecting two points on the line to calculate slop k ($k = \frac{T_2^2 - T_1^2}{L_2 - L_1}$), gravitational acceleration g can be obtained as

$$g = 4\pi^2\frac{L_2 - L_1}{T_2^2 - T_1^2} \tag{7}$$