2. Theory

1) <u>Dynamics Solution of Transverse Vibration of a Round Rod</u>

For transverse vibration of a slim rod, a dynamics solution can be derived by using Bernoulli-Euler beam model. This beam model ignores the influences of shear deformation and moment of inertia of the cross section related to the neutral axis. The Bernoulli-Euler model for force analysis is shown in Figure 1.

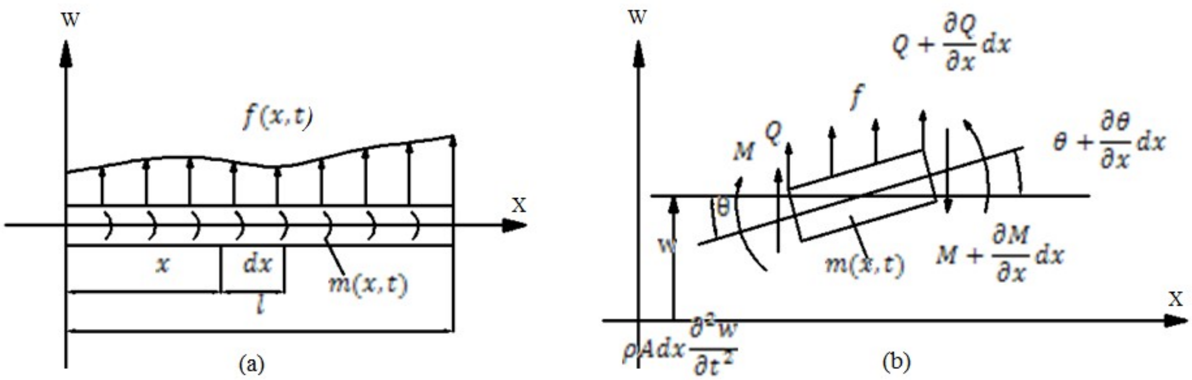

Figure 1 Vibration analysis of a bent round rod
(a) Force analysis of a round rod; (b) Force analysis of a micro-segment on the round rod.

Assume the length of the round rod is *l*. In Figure 1, *X* axis is along the central line of the round rod. *A, E,* ρ and *I* are respectively cross section, material's Young's modulus, density and inertia moment of corresponding to central axis. *W(x,t)* represents the transversal displacement of the cross section at coordinates *x* on the central axis at time *t*. *f(x,t)* and *m(x,t)* are respectively the received transversal force and force moment on unit length of the rod. Take a micro-segment of length *dx* for force analysis, where *Q(x,t)* and *M(x,t)* are respectively the shear force and bending moment on the cross section. $\rho A dx \frac{\partial^2 w}{\partial t^2}$ is the inertial force of the micro-segment. All forces and moments in Figure 1 are plotted in positive direction.

By Newton's second law, the bending vibration of a micro-segment of a round rod meets following relationship:

$$\begin{aligned} \rho A dx \frac{\partial^2 w(x,t)}{\partial t^2} &= Q(x,t) - \left[Q(x,t) + \frac{\partial Q(x,t)}{\partial x} dx \right] + f(x,t)dx \\ &= \left[f(x,t) - \frac{\partial Q(x,t)}{\partial x} \right] dx \end{aligned} . \quad (1)$$

Since the rotational inertia relating to central axis is ignored, the torque along rotational direction of the micro-segment is equilibrium. We have:

$$M(x,t) + Q(x,t)dx = M(x,t) + \frac{\partial M(x,t)}{\partial x} dx + m(x,t)dx . \quad (2)$$

Simply (2), we get:

$$Q(x,t) = \frac{\partial M(x,t)}{\partial x} + m(x,t)\,. \tag{3}$$

Substitute (3) into (1), we get:

$$\rho A \frac{\partial^2 w(x,t)}{\partial x^2} = f(x,t) - \left[\frac{\partial M^2(x,t)}{\partial x^2} + \frac{\partial m(x,t)}{\partial x} \right]. \tag{4}$$

Bending moment and deflection have following relationship:

$$M(x,t) = EI \frac{\partial^2 w(x,t)}{\partial x^2}\,. \tag{5}$$

Substituting (5) into (4), we get the bending vibration equation of Bernoulli-Euler beam model of a round rod:

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = f(x,t) - \frac{\partial m(x,t)}{\partial x}\,. \tag{6}$$

For a free vibration of the rod, we can make $f(x,t) = 0$, $m(x,t) = 0$ in equation (6). So we get the free vibration equation of a round rod:

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EI \frac{\partial^4 w(x,t)}{\partial x^4} = 0\,. \tag{7}$$

Using the method of separation of variables to solve equation (7), and letting $W(x,t) = W(x)q(t)$, we get:

$$\rho A W(x)\ddot{q}(t) + EIW^{(4)}(x)q(t) = 0\,. \tag{8}$$

i.e.:

$$\frac{EIW^{(4)}(x)}{\rho A W(x)} = -\frac{\ddot{q}(t)}{q(t)}\,. \tag{9}$$

In equation (9), the left side is a function of *x* and the right side is a function of *t*. *x* and *t* are independent. Therefore, both sides of above equation are equal to a constant, we make:

$$\frac{EIW^{(4)}(x)}{\rho A W(x)} = -\frac{\ddot{q}(t)}{q(t)} = \omega^2\,. \tag{10}$$

We further get:

$$\begin{cases} W^{(4)}(x) - \lambda^4 W(x) = 0 & (11.a) \\ \ddot{q}(t) + \omega^2 q(t) = 0 & (11.b) \end{cases}$$

Where:

$$\lambda^4 = \frac{\rho A}{EI}\omega^2\,. \tag{12}$$

Solving Equations (11) and (12), we get:

$$\begin{cases} W(x) = a_1 \cos\lambda x + a_2 \sin\lambda x + a_3 ch\lambda x + a_4 sh\lambda x & (13.a) \\ q(t) = b_1 \cos\omega t + b_2 \sin\omega t & (13.b) \end{cases}$$

For a rod with free ends, the bending moment and shear force are zero at both ends, i.e.

$$\begin{cases} M = EI \dfrac{\partial^2 W}{\partial x^2} = 0 \\ Q = EI \dfrac{\partial^3 W}{\partial x^3} = 0 \end{cases}$$

Therefore, we have:

$$\begin{cases} W''(0) = 0 & W'''(0) = 0 \\ W''(l) = 0 & W'''(l) = 0 \end{cases}. \qquad (14)$$

Solving (14), we get:

$$ch\lambda l \cos\lambda l = 1 . \qquad (15)$$

Solving (15), we get:

$$\lambda_0 l = 0, \lambda_1 l = 1.5060\pi, \lambda_2 l = 2.4997\pi, \lambda_3 l = 3.5004\pi, \Lambda \;, \lambda_n l \approx (n+1/2)\pi . \quad (16)$$

According to (12), above solutions λ_r have corresponding frequencies as follows:

$$f_r = \frac{(\lambda_r l)^2}{2\pi}\sqrt{\frac{EI}{\rho A l^4}}, \qquad r = 1,2,3,\Lambda \; . \qquad (17)$$

These are the frequencies corresponding to various vibration modes of the bending rod.

Frequency f_1 at r=1 corresponds to the fundamental (first order) vibration mode of bending rod of two ends free. The actual vibration is the superposition of all vibration modes. In general, the inherent frequency $f_{natural}$ of a system is closed to frequency f_1 of the first order mode, while the resonant frequency is determined by system inherent frequency $f_{natural}$ and damping β. In case of small damping, the inherent frequency $f_{natural}$ and the resonant frequency $f_{resonant}$ are approximately equal. The specific relationship is given as follows:

$$f_{resonant}^{\;2} = f_{natural}^{\;2} - \beta^2 . \qquad (18)$$

For this apparatus, the fundamental (first order) vibration is the main vibration, and its resonant frequency $f_{resonant}$ is approximately equal to f_1. Therefore, it is adequate to substitute f_1 in Eq. (17) with resonant frequency $f_{resonant}$ which can be measured using resonance method. Furthermore, Young's modulus of the material can be derived from Eq. (17).

Below, we find the solution of the main vibration mode $W_1(x)$. Substituting the above obtained $\lambda_1 l = 1.5060\pi$ into (13.a), we have:

$$W_1(\frac{x}{l}) = a_1 \cos(\lambda_1 l \cdot \frac{x}{l}) + a_2 \sin(\lambda_1 l \cdot \frac{x}{l}) + a_3 ch(\lambda_1 l \cdot \frac{x}{l}) + a_4 sh(\lambda_1 l \cdot \frac{x}{l}) . \qquad (19)$$

where coefficients a_1, a_2, a_3 and a_4 can be derived from Eq. (14) as:

$a_1 = a_3 = \cos\lambda_1 l - ch\lambda_1 l, \qquad a_2 = a_4 = \sin\lambda_1 l - sh\lambda_1 l$.

Let $W_1(\frac{x}{l}) = 0$, we get $\frac{x}{l} = 0.224$ or 0.776. The two points are the main vibration nodes, and are also approximately the resonant nodes of main vibration. At the two points, the amplitude of vibration displacement is always 0. By equation (19), the resonant amplitude along the rod of ends free can be plotted as shown in Figure 2.

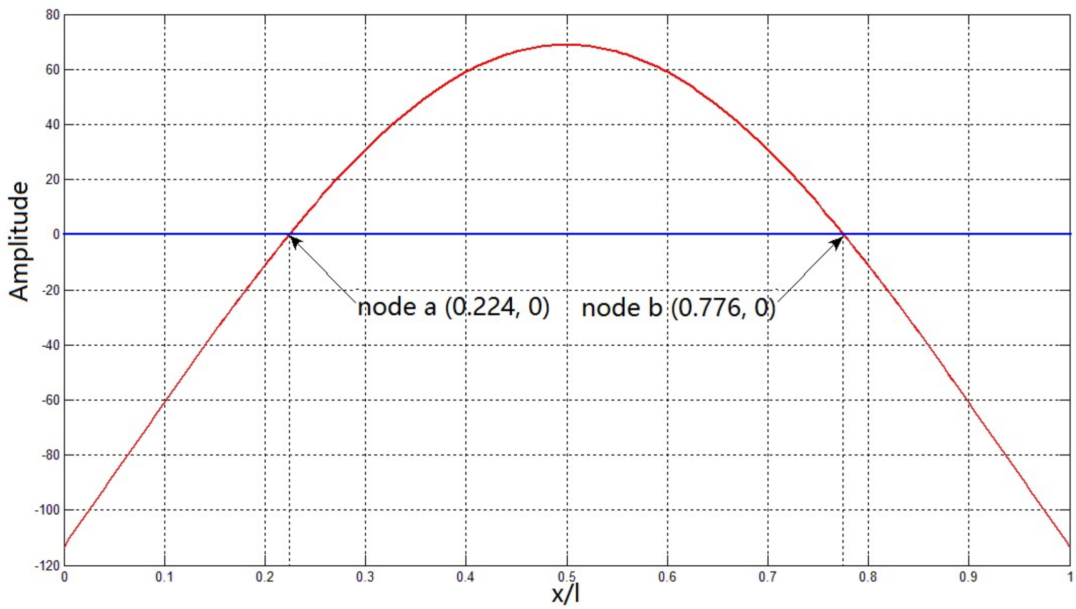

Figure 2 Resonant amplitude along round rod of ends free

From above we know the resonant frequency $f_{resonant}$ of a round rod is:

$$f_{resonant} = \frac{(\lambda_1 l)^2}{2\pi}\sqrt{\frac{EI}{\rho A l^4}} = \frac{(1.5060\pi)^2}{2\pi}\sqrt{\frac{EI}{\rho A l^4}} . \tag{20}$$

For a round rod, its cross section A can be acquired through the measurement of diameter. Length l can be easily measured by using a ruler. Density ρ can be achieved from dividing rod mass m (which can be weighted by a balance) with rod volume (which is Al). The inertia moment I of rod cross section relative to its central axis is $I = \frac{\pi}{64} d^2$. Equation (20) can be rewritten as:

$$E = 1.6067 \times \frac{m l^3 f_{resonant}{}^2}{d^4} . \tag{21}$$

For Eq. (21), the resonant frequency is measured by this Young's modulus apparatus. So, material's Young's modulus can be derived.

2) Method of Reducing Measurement Error

Since the above theory is based on Bernoulli-Euler beam model of ends free, the influences of shear deformation and rotational inertia of cross section relative to the central axis are ignored.

This condition may be strictly satisfied only at $d << l$. Therefore, a correction factor T_1 must be introduced into (21). The calculation formula of Young's modulus after correction is given as:

$$E = 1.6067 \times \frac{ml^3 f_{resonant}{}^2}{d^4} \times T_1 . \quad (22)$$

where the correction factor can be determined by the ratio of diameter to length of the round rod (i.e. *d/l*), as shown in Table 1.

Table 1 Comparison table between correction factor T_1 and ratio of diameter to length (*d/l*)

d/l	0.01	0.02	0.03	0.04	0.05	0.06	0.08	0.10
T_1	1.001	1.002	1.005	1.008	1.014	1.019	1.033	1.055

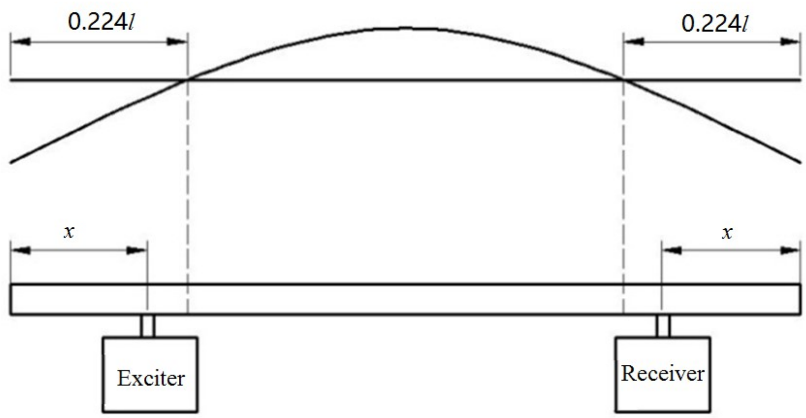

Figure 3 Schematic of the practical vibration measurement

At the same time, the beam model of ends free requires that the excitation point and the signal pick-up point must be the main vibration nodes of the beam. However, if placing the exciter and the signal pick-up receiver at the nodal points, the beam is hardly to vibrate. Therefore, through the measurement of resonant points with different *x* positions of the exciter and receiver, the rod resonant frequency with exciter and receiver exerting at nodal points can be derived by using extrapolation method. A schematic of the practical vibration measurement is shown in Figure 3.

Details of the extrapolation method are described as follows:

The length of the test rod sample is *l* (e.g. *l* =20 cm). The position of nodal point is 0.224*l* from each end according to the above theory. We measure the resonant frequencies of different *x* positions of exciter and receiver. The data is recorded in Table 2.

Table 2 Data of resonant frequencies at different positions

X (cm)	1.0	2.0	3.0	4.0	4.5	5.0	6.0	7.0
$f_{resonant}$ (Hz)	488.9	476.3	462.4	456.8	450.9	451.5	454.9	458.0

Plot a graph of position *x* and resonant frequency $f_{resonanct}$ on a coordinate paper using the data of Table 2 and connect them as a smooth curve, as shown in Figure 4. From the curve, we get the resonant frequency at nodal point 0.224*l* (i.e. *x*=4.48 cm) is 452.0 Hz. This point is the ideal resonant point of a rod of two ends free. Then, Young's modulus E of the rod material can be calculated using Eq. (22).

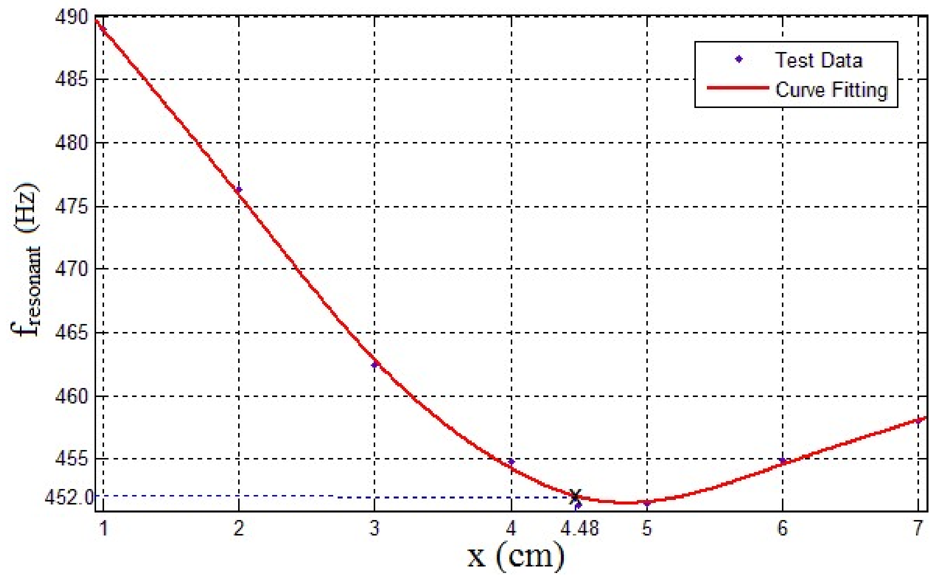

Figure 4 Fitted curve of measurement data.