## 2. Theory

### 1) Simulation of electrostatic field using current field

From electromagnetism theory, it is known that there is a similarity between a current field of steady current in a conductive material and a static electric field in dielectric media (or vacuum). In the no source region of a current field, current density vector *j* meets the conditions:

$$\oint j \times ds = 0 \quad \text{and} \quad \oint j \times dl = 0 \tag{1}$$

In the no source region of an electrostatic field, the electric field strength vector *E* meets the conditions:

$$\oint E \times ds = 0 \quad \text{and} \quad \oint E \times dl = 0 \tag{2}$$

From (1) and (2), it can be seen the current density vector *j* and the electric field strength vector *E* have the same mathematical form, so they have a similarity. If similar source distributions and boundary conditions are given, their mathematical models have same solution expressions. If two powered electrodes are placed in a conductive material, a current field will be generated in the material. There are some points having equal potential in the current field. These points can be measured and depicted in one plane, called equipotential plane. Similarly, there is also an equipotential surface in a static electric field. The electric field distribution is usually in three-dimensional space, while for the simulation on the surface of a conductive plate, the electric field distribution is measured in a horizontal plane. This way the equipotential surface becomes equipotential lines. As the current lines and the equipotential lines are orthogonal, the current lines can be drawn. The tangent direction at every point on a current line is the direction of the electric field strength *E* at corresponding point. Therefore, an electrostatic field distribution can be simply represented by equipotential lines and current lines.

To avoid an effect on the current line distribution when measuring equipotential lines in a current field, the measuring circuit cannot draw any current. Therefore, the measurement must use a voltmeter of high resistance or a balance electric bridge.

### 2) Electrostatic field and current field of coaxial cable electrode

Figure 1 shows a few examples of electrostatic field simulations. Figure 1(a) is the case of a coaxial cable electrode. Fix the coaxial electrode onto the surface of the conductive glass plate (with good contact) and apply a voltage $V_0$ (A is positive and B is ground). The current in conductive glass will form a stable constant current field along radial direction axially symmetrical from A to B. This simulated current field is similar to a static electric field. Their similarity can be analyzed as follows.

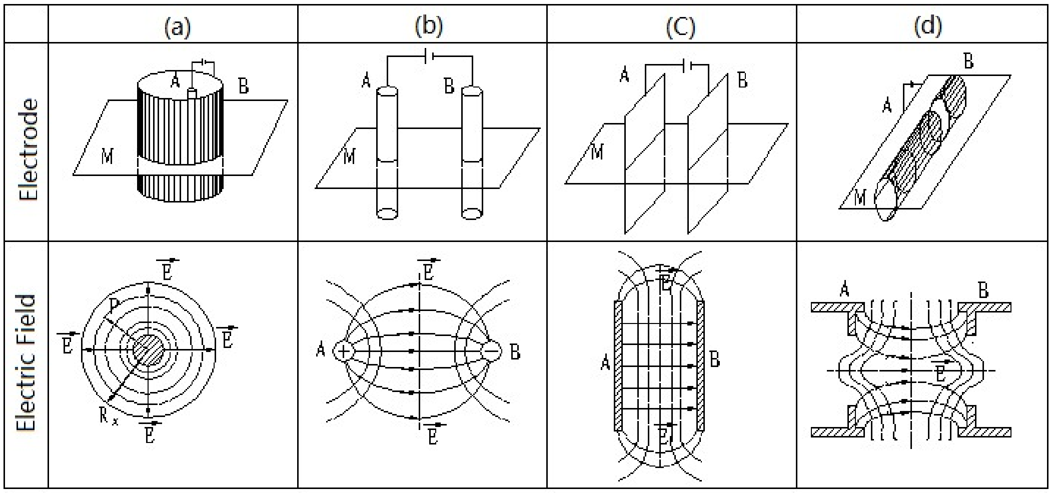


Figure 1 Examples of electrode patterns and corresponding electrostatic field equipotential lines distributions: (a) coaxial cable, (b) parallel-wire, (c) parallel-plane and (d) focusing electrode

### (i) Electrostatic field

Figure 2 (a) shows the structure of a long coaxial cable. There is a long cylindrical conductor of radius $r_1$ (electrode A) and a long tubular conductor of radius $r_2$ (electrode B) in vacuum. Their centers are coincidence. Assume the potentials of A and B are $V_A=V_1=V_0$ and $V_B=0$ (grounded), respectively, with equal but opposite charges, there will be an electric field between the two electrodes. Due to the symmetry, the electric lines will be uniformly distributed in any cross sections as seen in Fig. 2 (b). The equipotential surfaces of the electric field consist of series coaxial tubular surfaces. Electric lines and equipotential lines are orthogonal. The equipotential lines are closed lines, while the electric lines have both terminals that originate from positive charges and end at negative charges. For the metal central cylinder, the inside electric field intensity is zero and charges distribute on its surface. Electric lines start from the central cylinder and terminate at the inner surface of the tube wall.

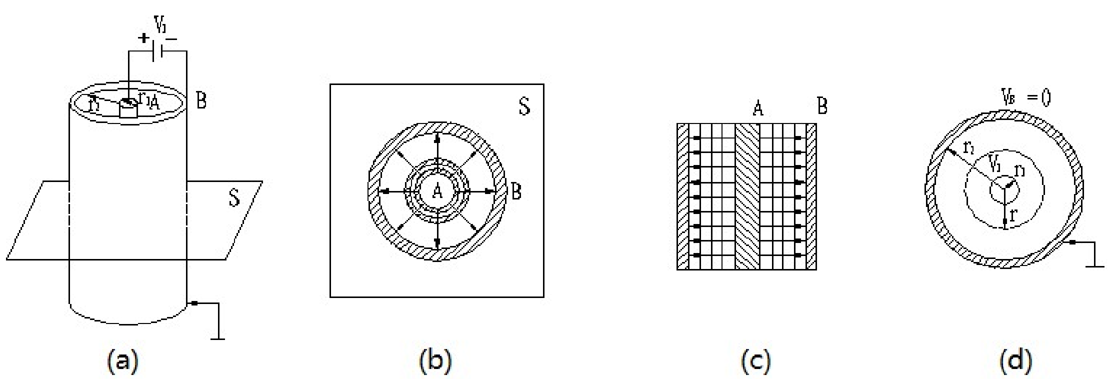


Figure 2 Electric field of a long coaxial electrode: (a) electrode structure, (b) electric line distribution in the perpendicular cross section, (c) electric line distribution in the longitudinal cross section, and (d) schematic for calculation

To calculate the electrostatic field between A and B, we take a perpendicular cross section as shown in Figure 2 (d), and assume the charge amount on the inside circle and the outside circle are $+\in$ and $-\in$, respectively. Draw a circle between the two circles with radius r and assume the electric field strength on this circle is $E$, from Gaussian theorem, we have $\in = 2r\varepsilon_o E$, i.e.

$$E = -dv/dr = \in/2\pi\varepsilon_o r \tag{3}$$

From (3), we get: $Vr = -\int E dr = -\in/2\pi\varepsilon_o \int dr/r = -K\int dr/r$, therefore,

$$Vr = -K\ln r + C \tag{4}$$

where $K = \in/2\pi\varepsilon_o$. Applying boundary conditions: $Vr=V_1=V_0$ at $r=r_1$ and $Vr=V_2=0$ at $r=r_2$, we get the solution of equation (4) as:

$$Vr = V_0\cdot[\ln(r/r_2)/\ln(r_1/r_2)] \tag{5}$$

From (5), it can be seen that Vr and lnr have a linear relationship, and the relative potential $Vr/V_0$ is only a function of coordinate r.

**(ii) Current field**

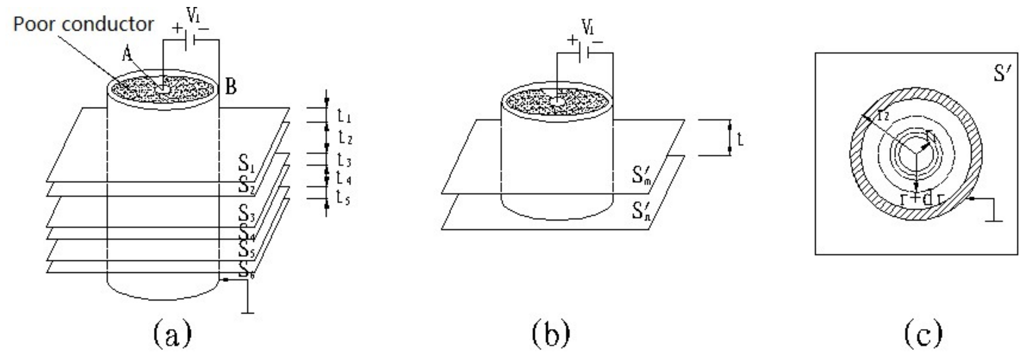


Figure 3 Current field of a long coaxial cable: (a) cable structure, (b) current line distribution in the perpendicular cross section, (c) schematic for calculation

As seen in Fig. 3, the space between electrodes A and B is filled with uniform conductive material to simulate a field that is similar to a static electric field. This device is called "simulation model". By measuring the simulated field, we can get the distribution of an electrostatic field.

To calculate the potential difference of a current field, first calculate the resistance between the two cylindrical surfaces, then calculate the current, and finally calculate the potential difference between any two points. Assume the thickness of the thin conductive medium layer (such as conductive glass) is t and the resistivity is p, the resistance of the ring between any radius r and r+dr circle is:

$$dR = p\cdot dr/s = pdr/2\pi t\cdot dr/r \tag{6}$$

Do integral operation from radius r to radius $r_2$ using (6), the total resistance is:

$$R_{rr2} = p/2\pi t \int^{r2} r\cdot dr/r = p/2\pi t\cdot \ln(r_2/r) \tag{7}$$

Similarly, the total resistance between radius $r_1$ and $r_2$ is:

$$R_{12} = p/2\pi\, t\cdot\int^{r2} r\cdot dr/\, r = p/2\pi t\cdot \ln(r_2/r_1) \tag{8}$$

Therefore, the current from the inner cylindrical surface to the outer cylindrical surface is:

$$I_{12} = V_0/R_{12} = 2\pi t/p\, \ln(r_2/r_1)\cdot V_0 \tag{9}$$

The potential from the outer cylindrical surface ($V_2$=0) to radius r is:

$$V_r = I_{12} \cdot Rr\ r_2 = Rr\ r_2 / R_{12} \cdot V_0 \qquad (10)$$

Substitute (7) and (8) into (10), we have:

$$V_r = V_0 \cdot (\ln\ r_2/r)/(\ln r_2/r_1) = V_0 \cdot (\ln r/r_2)/(\ln r_1/r_2) \qquad (11)$$

Compare (11) with (5), it is seen that the potential distributions of a simulated current field and a static electric field are identical.

The above discussion is an example of same potential distribution of an electrostatic field and a current field with same boundary conditions. It is difficult to obtain the electrostatic field of a complex electrode pattern using analytic method. In this case, the advantage is obvious to use a current field to simulate an electrostatic field.

**(iii) Electric field of a pair of long parallel lines**

As shown in Figure 4 (a), two long parallel cylindrical conductive lines A and B have opposite potential $+V_1$ and $-V_1$, respectively. Due to the symmetry, there exists many electric field planes perpendicular to the wires in the static electric field. Plane *S* is an example as shown in Figure 4 (a).

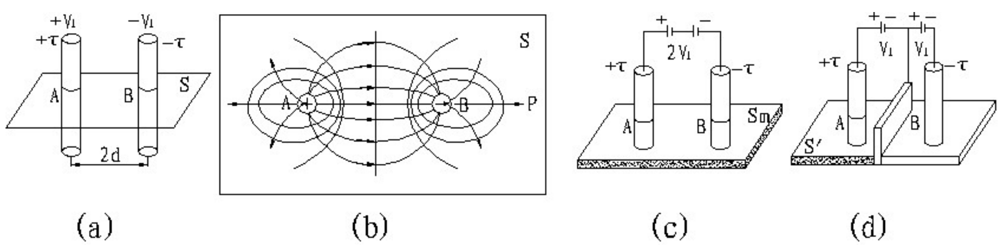


Figure 4 Electric field of a pair of long parallel lines

Use a uniformly conductive medium to fill the entire electric field space and connect a battery of potential $2V_1$ between electrodes A and B, to make a simulation model as shown in Figure 4(c). The electric field distribution in the poor conductor will not change when a stable current exists.

In the electric field of the long parallel wires, there exists a flat equipotential surface, i.e. the plane at the middle point of the connection line of the two wires, so that the simulation model can be simplified. Insert a metal plate at the middle plane, the metal plate will have same potential with the middle point of the $2V_1$ battery. Connect the metal plate with the equipotential point using a conductive wire to get the status as shown in Figure 4(d). Now, the current states on both sides of the metal plate will be same as that shown in Figure 4(c) and is left-right symmetric. In experiment, once the electric field of one half space is measured, the other half will be known.